Abscissas and Weight Coefficients for Lobatto Quadrature

By H. H. Michels

1. Introduction. Recently, the numerical evaluation of certain collision integrals was studied using several different mechanical quadrature formulas, including Gaussian quadrature of high order [1, 2] and various Newton-Cotes formulas. It was found that high accuracy could not easily be obtained, owing to the particular behavior of the integrand at the end points of integration, and it seemed likely that a "closed" type Gaussian formula of high order might be more efficient for this particular application.

The existence of Gaussian-type quadrature formulas with one or more prescribed abscissas has been investigated by Lobatto [3] and Radau [4]. For the case where both ends of the integration interval are preassigned (Lobatto quadrature), the free abscissas and the corresponding weight coefficients have been evaluated by Radau [5] up to order 11. More recently abscissas and weights for Lobatto quadrature have been reported by Rabinowitz [6] for selected odd order up to 65. In some cases, however, an even-order quadrature formula may be desired and the results for such formulas of high order are reported in this communication.

2. Method of Computation. We are concerned with the Lobatto quadrature formulas of order n normalized by a change of variables to the interval $(-1, 1)$

$$\int_{-1}^{+1} f(x)\,dx = H_1 f(-1) + \sum_{k=2}^{n-1} H_k f(x_k) + H_n f(+1). \tag{1}$$

Formula (1) is exact for all polynomials $f(x)$ of degree $\leqq 2n - 3$, whereas Gaussian quadrature rules are exact for degree $\leqq 2n - 1$. However, if the function $f(x)$ is zero at both ends of the integration interval, only $r - 2$ ordinates are involved in the calculation and a higher effective degree of precision is obtainable than if an open Gaussian type formula is used. The free abscissas x_k $(k = 2, 3, \cdots, n - 1)$ are the zeros of the first derivative of the Legendre polynomial of order $n - 1$

$$P'_{n-1}(x_k) = 0. \tag{2}$$

The corresponding weight coefficients H_k can be found from the expression

$$H_k = \frac{2}{n(n-1)[P_{n-1}(x_k)]^2} \tag{3}$$

where $P_{n-1}(x_k)$ is the normalized Legendre polynomial of order $n - 1$. The weights corresponding to the fixed abscissas at $x = \pm 1$ are found to be

$$H_{-1} = H_{+1} = \frac{2}{n(n-1)}. \tag{4}$$

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A first approximation to the zeros of the derivative of the Legendre polynomial $P_n'(x)$ can be obtained in several ways. It can be shown [7] that

$$\lim_{n\to\infty}\left\{n^{-m}P_n^{\;m}\left[\cos\left(\frac{x}{n}\right)\right]\right\} = J_m(x). \tag{5}$$

Since the zeros of $P_n'(x)$ are the same as those of the associated Legendre polynomial $P_n^{\;1}(x)$ through the relation

$$P_n^{\;1}(x) = (x^2 - 1)^{1/2}P_n'(x) \tag{6}$$

equation (5) can be used to relate the zeros of $P_n^{\;1}(x)$ to the successive zeros of the Bessel function $J_1(x)$.

A better approximation to the zeros of $P_n'(x)$ can be obtained by making use of the inequalities derived by Szegö [8] for the zeros of the generalized Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. An examination of the upper and lower bounds of the zeros of $P_{n-1}'(x)$ showed that two or three decimal places could be established using the relation

$$x_{n,k} = \cos\left\{\frac{j_{1,k}}{\left[(n - 1/2)^2 + \left(\frac{\pi^2 - 4}{4\pi^2}\right)\right]^{1/2}}\right\} \tag{7}$$

where $j_{1,k}$ are the successive zeros of the Bessel function $J_1(x)$. These initial approximations to the roots were improved using Newton-Raphson iteration

$$(x_{n,k})_{i+1} = (x_{n,k})_i - \frac{P_{n-1}'(x_{n,k})_i}{P_{n-1}''(x_{n,k})_i}. \tag{8}$$

The Legendre polynomials and their derivatives were computed using the recursion formulas

$$P_{n+1}(x) = \left(\frac{2n+1}{n+1}\right) x\, P_n(x) - \left(\frac{n}{n+1}\right) P_{n-1}(x) \qquad P_0(x) = 1; \quad P_1(x) = x \tag{9}$$

$$P_{n+1}'(x) = \left(\frac{2n+1}{n}\right) x\, P_n'(x) - \left(\frac{n+1}{n}\right) P_{n-1}'(x) \qquad P_0'(x) = 0; \quad P_1'(x) = 1 \tag{10}$$

$$P_{n+1}''(x) = \left(\frac{2n+1}{n-1}\right) x\, P_n''(x) - \left(\frac{n+2}{n-1}\right) P_{n-1}''(x) \qquad P_1''(x) = 0; \quad P_2''(x) = 3. \tag{11}$$

The weight coefficients were computed directly using equations (3) and (4).

3. Results. Abscissas and weights for Lobatto quadrature are presented in Table I for order $n = 3(1)16, 24, 32, 40, 48, 64, 80, 96$. All computations were performed on an IBM 7090 digital computer using extended precision routines. The tolerance for iteration on the roots was set at 1×10^{-22}. Several hand calculations of the roots and weight coefficients were performed. Complete agreement

TABLE I

Abscissas and Weight Coefficients for Lobatto Quadrature

Abscissas			Weights	
		$n = 3$		
1.0000000000	0000000000		0.3333333333	3333333333
0.0000000000	0000000000		1.3333333333	3333333333
		$n = 4$		
1.0000000000	0000000000		0.1666666666	6666666667
0.4472135954	9995793928		0.8333333333	3333333333
		$n = 5$		
1.0000000000	0000000000		0.1000000000	0000000000
0.6546536707	0797714380		0.5444444444	4444444444
0.0000000000	0000000000		0.7111111111	1111111111
		$n = 6$		
1.0000000000	0000000000		0.0666666666	6666666667
0.7650553239	2946469285		0.3784749562	9784698032
0.2852315164	8064509631		0.5548583770	3548635302
		$n = 7$		
1.0000000000	0000000000		0.0476190476	1904761905
0.8302238962	7856692987		0.2768260473	6156594801
0.4688487934	7071421380		0.4317453812	0986262342
0.0000000000	0000000000		0.4876190476	1904761905
		$n = 8$		
1.0000000000	0000000000		0.0357142857	1428571429
0.8717401485	0960661534		0.2107042271	4350603938
0.5917001814	3314230214		0.3411226924	8350436476
0.2092992179	0247886877		0.4124587946	5870388157
		$n = 9$		
1.0000000000	0000000000		0.0277777777	7777777778
0.8997579954	1146015731		0.1654953615	6080552505
0.6771862795	1073775345		0.2745387125	0016173528
0.3631174638	2617815871		0.3464285109	7304634512
0.0000000000	0000000000		0.3715192743	7641723356
		$n = 10$		
1.0000000000	0000000000		0.0222222222	2222222222
0.9195339081	6645881383		0.1333059908	5107011113
0.7387738651	0550507500		0.2248893420	6312645212
0.4779249498	1044449566		0.2920426836	7968375788
0.1652789576	6638702463		0.3275397611	8389745666
		$n = 11$		
1.0000000000	0000000000		0.0181818181	8181818182
0.9340014304	0805913433		0.1096122732	6699486446
0.7844834736	6314441862		0.1871698817	8030520411
0.5652353269	9620500647		0.2480481042	6402831404
0.2957581355	8693939143		0.2868791247	7900808868
0.0000000000	0000000000		0.3002175954	5569069379
		$n = 12$		
1.0000000000	0000000000		0.0151515151	5151515152
0.9448992722	2288222341		0.0916845174	1319613067
0.8192793216	4400667835		0.1579747055	6437011517
0.6328761530	3186067766		0.2125084177	6102114536
0.3995309409	6534893226		0.2512756031	9920128029
0.1365529328	5492755486		0.2714052409	1069617700
		$n = 13$		
1.0000000000	0000000000		0.0128205128	2051282051
0.9533098466	4216391190		0.0778016867	4681892779

Abscissas			Weights	
0.8463475646	5187231687		0.1349819266	8960834912
0.6861884690	8175742607		0.1836468652	0355009201
0.4829098210	9133620175		0.2207677935	6611008609
0.2492869301	0623999257		0.2440157903	0667635646
0.0000000000	0000000000		0.2519308493	3344673604
		$n = 14$		
1.0000000000	0000000000		0.0109890109	8901098901
0.9599350452	6726090135		0.0668372844	9768128463
0.8678010538	3034725100		0.1165866558	9871165154
0.7288685990	9132614059		0.1600218517	6295214241
0.5506394029	2864705532		0.1948261493	7341611864
0.3427240133	4271284504		0.2191262530	0977075487
0.1163318688	8370386766		0.2316127944	6845705889
		$n = 15$		
1.0000000000	0000000000		0.0095238095	2380952381
0.9652459265	0383857280		0.0580298930	2860124910
0.8850820442	2297629883		0.1016600703	2571806760
0.7635196899	5181520070		0.1405116998	0242810946
0.6062532054	6984571112		0.1727896472	5360094905
0.4206380547	1367248092		0.1969872359	6461335609
0.2153539553	6379423823		0.2119735859	2682092013
0.0000000000	0000000000		0.2170481163	4881564951
		$n = 16$		
1.0000000000	0000000000		0.0083333333	3333333333
0.9695680462	7021793295		0.0508503610	0591990540
0.8992005330	9347209299		0.0893936973	2593080099
0.7920082918	6181506393		0.1242553821	3251409835
0.6523887028	8249308947		0.1540269808	0716428081
0.4860594218	8713761178		0.1774919133	9170412530
0.2998304689	0076320810		0.1936900238	2520358432
0.1013262735	2194944784		0.2019583081	7822987149
		$n = 24$		
1.0000000000	0000000000		0.0036231884	0579710145
0.9867305535	0516088355		0.0222368534	6471120899
0.9557482209	2988635803		0.0396316813	3346780947
0.9077056751	1350652200		0.0563098487	2464619902
0.8434640701	5487204062		0.0719818620	5529398222
0.7641704824	2049330779		0.0863690299	6792906822
0.6712401052	6412869984		0.0992148276	8408358741
0.5663313579	7929531219		0.1102900868	9296860411
0.4513163732	1432261825		0.1193971937	0249131903
0.3282476133	7551091203		0.1263736420	2802080013
0.1993212533	9083266724		0.1310949418	7360394235
0.0668379937	3722857811		0.1334768438	6698637760
		$n = 32$		
1.0000000000	0000000000		0.0020161290	3225806452
0.9926089339	7276135937		0.0123981065	0137384379
0.9752946904	8270922806		0.0221995528	8929196462
0.9482848384	1723237808		0.0317751354	1091546578
0.9118499390	6373190407		0.0410342015	8606272333
0.8663524760	1267551983		0.0498852713	3622120701
0.8122447317	7744234455		0.0582404972	4805586955
0.7500644939	3667479772		0.0660168772	5715454393
0.6804297556	1555081594		0.0731371396	0267903264
0.6040325871	4842112614		0.0795305256	9210625229
0.5216322628	8156529061		0.0851334979	4966823053
0.4340477172	0184693960		0.0898903729	5735783307

Abscissas	Weights
0.3421494065 3888148625	0.0937538755 4681381357
0.2468506588 5020530442	0.0966856089 4800260056
0.1490985968 1364749491	0.0986564365 4076177717
0.0498647250 4659325231	0.0996467715 0127677764
$n = 40$	
1.0000000000 0000000000	0.0012820512 8205128205
0.9952979292 4434889690	0.0078910115 8860061376
0.9842662807 1750335473	0.0141593075 4991977315
0.9670100764 8798852065	0.0203347590 6338716185
0.9436397649 4360164257	0.0263811906 5314148626
0.9143033396 9020945107	0.0322607179 2711739549
0.8791863434 7933983789	0.0379362437 0070844699
0.8385108227 7810644074	0.0433719081 9475798101
0.7925339526 0155188681	0.0485333538 4591432514
0.7415464191 4738441749	0.0533879519 7149418296
0.6858705850 8431371383	0.0579050119 8178608340
0.6258584527 5525751340	0.0620559764 7570952497
0.5618894392 9472264877	0.0658146022 2289590266
0.4943679781 2525360665	0.0691571262 7608113446
0.4237209621 5555098475	0.0720624163 0205429522
0.3503950449 1418087798	0.0745121042 3538933858
0.2748538167 1432436652	0.0764907024 3339650625
0.1975748737 1891077184	0.0779857016 0868058073
0.1190467984 4497109352	0.0789876499 2536434419
0.0397660708 0218190015	0.0794902127 6154964087
$n = 48$	
1.0000000000 0000000000	0.0008865248 2269503546
0.9967477813 3985746440	0.0054591926 0024811922
0.9891114700 1363572789	0.0098069319 7890032805
0.9771488468 9083677288	0.0141094906 0548871324
0.9609131535 0638159011	0.0183500364 7521908147
0.9404755493 3508129520	0.0225102637 3693608450
0.9159254499 7624642008	0.0265720325 9091131712
0.8873702243 1822009720	0.0305175976 2130364253
0.8549347448 3877407027	0.0343297117 0067383698
0.8187608474 8560586870	0.0379917079 2672815653
0.7790067135 9984382197	0.0414875745 1884729112
0.7358461791 1284836876	0.0448020255 6407161048
0.6894679748 1109071037	0.0479205681 5673550066
0.6400749012 6595713783	0.0508295659 0594469726
0.5878829421 4548518103	0.0535162986 2867493123
0.5331203198 3277696111	0.0559690180 0514883558
0.4760264975 0285707802	0.0581769989 6927463166
0.4168511320 3270266991	0.0601305866 1684402305
0.3558529823 2890604382	0.0618212384 2997936681
0.2932987778 4975926536	0.0632415616 3503798566
0.2294620522 7124457033	0.0643853455 3160117653
0.1646219473 9809062748	0.0652475886 5172582358
0.0990619925 5075011235	0.0658245206 3100801710
0.0330688647 6615291109	0.0661136186 9600179419
$n = 64$	
1.0000000000 0000000000	0.0004960317 4603174603
0.9981798715 0216321518	0.0030560082 4491249038
0.9939027267 0305729237	0.0054960162 0381715690
0.9871926766 0274024265	0.0079212897 9004663404
0.9780666628 3139607396	0.0103270023 6681532846
0.9665471103 6909923352	0.0127073991 9745473520

Abscissas		Weights	
0.9526622357	8866291546	0.0150566839	8796144273
0.9364460274	7563416245	0.0173691163	8454218159
0.9179381735	1028163083	0.0196390407	2324171838
0.8971839678	4585004284	0.0218609035	1151806011
0.8742342006	5762749177	0.0240292681	4402382671
0.8491450345	4299098500	0.0261388286	1433843775
0.8219778673	0751705008	0.0281844226	6584851747
0.7927991818	2620813735	0.0301610444	9908945068
0.7616803834	0811996779	0.0320638570	5772702451
0.7286976250	8883693957	0.0338882038	8412539761
0.6939316212	9070484081	0.0356296205	2448948628
0.6574674503	1297650421	0.0372838454	5980117259
0.6193943461	3843154754	0.0388468305	3780773670
0.5798054800	6771842373	0.0403147508	8156023712
0.5387977327	1680043899	0.0416840142	5080195219
0.4964714569	3605775350	0.0429512698	3360181861
0.4529302322	3158118199	0.0441134164	4689247092
0.4082806112	8985404453	0.0451676101	2594770204
0.3626318592	2626182369	0.0461112710	8428905945
0.3160956861	9562580640	0.0469420900	2702831645
0.2687859740	1917000399	0.0476580338	0222063679
0.2208184974	9695350321	0.0482573503	7641454890
0.1723106410	8779297132	0.0487385731	2223318512
0.1233811116	5002798616	0.0491005244	0750130793
0.0741496479	4611591873	0.0493423184	7713957394
0.0247367276	2195872850	0.0494633636	2077664644
	$n = 80$		
1.0000000000	0000000000	0.0003164556	9620253165
0.9988386765	3092507835	0.0019500843	5097755934
0.9961086610	7990519938	0.0035089042	0102901840
0.9918229201	8121586819	0.0050614259	4958905768
0.9859884777	6335317952	0.0066059324	9516823379
0.9786145023	1691623735	0.0081400967	2701197166
0.9697125241	8079561736	0.0096615416	7369727760
0.9592964489	5627958346	0.0111678973	4774327942
0.9473825428	4356652296	0.0126568138	2026628430
0.9339894093	4788020310	0.0141259672	1119237429
0.9191379609	9088049972	0.0155730640	3491764695
0.9028513869	7154153396	0.0169958450	4087065425
0.8851551171	0238834635	0.0183920888	4689349982
0.8660767821	7504981912	0.0197596154	5504496628
0.8456461708	5578157538	0.0210962896	7828148968
0.8238951831	9510250605	0.0224000244	8605002555
0.8008577808	3200733070	0.0236687842	6930063219
0.7765699339	7441852493	0.0249005880	2244976823
0.7510695652	4070981102	0.0260935124	3860403063
0.7243964904	5110723775	0.0272456949	1386453736
0.6965923564	6205299908	0.0283553364	5636954355
0.6677005761	4097178467	0.0294207044	9572629427
0.6377662605	8319665727	0.0304401355	8855621511
0.6068361486	7703163943	0.0314120380	1599442732
0.5749585341	2701742034	0.0323348942	6912755883
0.5421831900	4940380757	0.0332072634	1851378337
0.5085612912	5760347898	0.0340277833	6410102158
0.4741453343	5899572548	0.0347951729	6204047278
0.4389890557	8785790006	0.0355082340	2508140861
0.4031473479	0241926342	0.0361658531	9342834538
0.3666761732	7705082998	0.0367670036	7314253757

Abscissas			Weights	
0.3296324773	2342034030		0.0373107468	3937560243
0.2920740993	7704882920		0.0377962337	0193349881
0.2540596823	8810009988		0.0382227062	3088360543
0.2156485813	5741284831		0.0385894985	4013587066
0.1769007706	6074369751		0.0388960379	2715055300
0.1378767504	0592486577		0.0391418457	6714956376
0.0986374519	6914984692		0.0393265382	6043549120
0.0592441428	5788193249		0.0394498270	3165166113
0.0197583310	4893162978		0.0395115195	8004770610
		$n = 96$		
1.0000000000	0000000000		0.0002192982	4561403509
0.9991951753	7692604333		0.0013515349	0565556724
0.9973028330	1700828646		0.0024325786	0301058480
0.9943310519	9061228080		0.0035104223	7502451778
0.9902832781	5433043846		0.0045843897	8477942644
0.9851639321	2661401114		0.0056533772	3637145606
0.9789785649	7778433333		0.0067162407	8972019234
0.9717338739	8905729625		0.0077718342	1191470895
0.9634377002	8792649710		0.0088190167	1167706097
0.9540990218	2054016010		0.0098566557	8151741141
0.9437279441	6218866932		0.0108836289	1988202822
0.9323356897	9926625583		0.0118988250	2549847790
0.9199345860	8504168194		0.0129011456	7292458482
0.9065380519	4989011153		0.0138895063	3390918365
0.8921605834	0720878924		0.0148628375	6675032461
0.8768177378	8235794840		0.0158200861	8163132671
0.8605261173	8703298179		0.0167602163	8464160450
0.8433033505	5998190621		0.0176822109	0106401665
0.8251680735	9491829583		0.0185850720	7759767582
0.8061399100	7704236359		0.0194678229	6277537046
0.7862394497	5042948018		0.0203295083	6464755771
0.7654882262	3952078195		0.0211691958	8472200913
0.7439086937	4898009871		0.0219859769	2711685636
0.7215242027	6722755296		0.0227789676	8188025685
0.6983589748	0000278398		0.0235473100	8144007028
0.6744380761	6133855505		0.0242901727	2916563323
0.6497873908	5033178220		0.0250067517	9904768294
0.6244335925	4307948931		0.0256962719	0552992516
0.5984041157	3009891832		0.0263579869	4255565597
0.5717271260	3047185738		0.0269911808	9092463744
0.5444314897	1484151328		0.0275951685	9308872911
0.5165467424	7024456131		0.0281692964	9454935815
0.4881030574	4058016068		0.0287129433	5105561178
0.4591312125	7730066577		0.0292255209	0083844395
0.4296625573	3565453103		0.0297064745	0115411602
0.3997289787	5251961757		0.0301552837	2844846757
0.3693628669	4253394460		0.0305714629	4149287708
0.3385970800	4986013077		0.0309545618	0688276681
0.3074649086	9350863338		0.0313041657	8633018045
0.2760000399	4469276349		0.0316198965	8522326545
0.2442365208	7519472791		0.0319014125	6196737563
0.2122087217	1618606717		0.0321484090	9766492590
0.1799512986	6736731540		0.0323606189	2573403155
0.1474991563	9667004105		0.0325378124	2110930124
0.1148874102	7109922607		0.0326797978	4871187911
0.0821513483	5958483863		0.0327864215	7091989724
0.0493263932	4895813657		0.0328575682	1381485879
0.0164480637	1437043510		0.0328931607	9202407467

with the machine results was found in all cases to 21 decimals. In addition, the following relation was used as a check on the accuracy of the results

$$\sum_{k=1}^{n} H_k = 2. \tag{12}$$

Equation (12) was satisfied to within 2 units in the 21st decimal place for all cases reported here.

In the table only the positive abscissas are reported since all abscissas and weights are symmetric with $x_{-k} = -x_k$ and $H_{-k} = H_k$.

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